



Bondal's conjecture in dimension five

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Foliation, Fibration and Moduli

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Holomorphic Poisson structures

Let X be a complex manifold.

Definition

A **holomorphic Poisson structure** on X is a bivector field

$$\sigma \in H^0(X, \wedge^2 \mathcal{T}_X) \quad \text{with} \quad [\sigma, \sigma] = 0,$$

where $[-, -]$ is the Schouten bracket on polyvector fields.

Equivalently, $\{f, g\} := \sigma(df, dg)$ is a Lie bracket on $\mathcal{O}_X$ satisfying Leibniz.

Contraction with σ gives the **anchor**

$$\sigma^\sharp: \Omega_X^1 \longrightarrow \mathcal{T}_X, \quad \alpha \mapsto \iota_\alpha \sigma.$$

At each point x , $\sigma_x^\sharp$ is a skew-symmetric map $\Omega_{X,x}^1 \rightarrow \mathcal{T}_{X,x}$; its rank is **even** and it may jump.

The characteristic foliation

$[\sigma, \sigma] = 0 \implies \sigma^\sharp(\Omega_X^1) \subset \mathcal{T}_X$ is closed under the Lie bracket.

Definition

$\mathcal{F}_\sigma \subset \mathcal{T}_X$ is the **saturation** of $\sigma^\sharp(\Omega_X^1)$: a holomorphic foliation on X .

- $\sigma^\sharp(\Omega_X^1)$ itself is not saturated: its rank drops on the degeneracy loci.
- Leaves of $\mathcal{F}_\sigma$ are in general **unions** of symplectic leaves of σ ; the two agree only on the dense open set where σ has maximal rank.

Write $2r$ for the generic rank of σ ; then $\mathcal{F}_\sigma$ has dimension $2r$.

Symplectic leaves

On $X^\circ = \{\text{rank } \sigma_x^\sharp = 2r\}$, open and dense, the foliation $\mathcal{F}_\sigma|_{X^\circ}$ is regular and $\sigma^\sharp$ induces an isomorphism from $\Omega_{\mathcal{F}_\sigma}^1$ to $\mathcal{T}_{\mathcal{F}_\sigma}$. Its inverse is a leafwise holomorphic symplectic form.

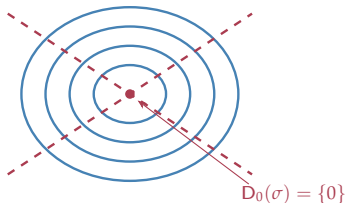
Example. On $X = \mathbb{C}^3$ take

$$\sigma = x \partial_y \wedge \partial_z + y \partial_z \wedge \partial_x + z \partial_x \wedge \partial_y,$$

the linear Poisson structure on $\mathfrak{sl}_2^*$. Its Casimir is $q = x^2 + y^2 + z^2$, and the symplectic leaves are

- the smooth quadrics $\{q = c\}, c \neq 0$;
- the punctured cone $\{q = 0\} \setminus \{0\}$;
- the origin.

So $D_0(\sigma) = \{0\}$ and $r = 1$.



Foliations and Poisson structures

Rank at most 2

Let $\mathcal{F} \subset \mathcal{T}_X$ be a foliation of dimension 2. Then $\wedge^2 \mathcal{T}_{\mathcal{F}} = \det \mathcal{T}_{\mathcal{F}}$ and every

$$\sigma \in H^0(X, \det \mathcal{T}_{\mathcal{F}}) \subset H^0(X, \wedge^2 \mathcal{T}_X)$$

is Poisson, since $[\sigma, \sigma] \in H^0(\wedge^3 \mathcal{T}_{\mathcal{F}}) = 0$. Conversely every σ of generic rank 2 arises this way, with $\mathcal{F} = \mathcal{F}_{\sigma}$.

- $\dim X = 2$: every $\sigma \in H^0(X, -K_X)$ is Poisson, and $D_0(\sigma)$ is its divisor of zeros.
- $\dim X = 2n$, σ of maximal rank $2n$: $\sigma^n \in H^0(X, -K_X)$, and $D_{2n-2}(\sigma)$ is its divisor of zeros.
- $\dim X = 2n + 1$, σ of **maximal rank $2n$** : then $\mathcal{F}_{\sigma}$ is a **codimension-one foliation**.

Degeneracy loci

Definition

For $k \geq 0$, the **$2k$ -th degeneracy locus** is

$$D_{2k}(\sigma) := \{x \in X : \text{rank } \sigma_x^\sharp \leq 2k\},$$

with its scheme structure as the zero locus of the Pfaffian $\sigma^{k+1} \in H^0(X, \wedge^{2k+2} \mathcal{T}_X)$.

$$\begin{array}{ccccccc} \{\sigma = 0\} & & & & & & \{\sigma^r = 0\} \\ D_0 & \longrightarrow & D_2 & \longrightarrow & D_4 & \longrightarrow & \cdots \longrightarrow D_{2r-2} \longrightarrow X \end{array}$$

- Each $D_{2k}(\sigma)$ is a **strong Poisson subscheme**: it is preserved by every ζ with $\mathcal{L}_\zeta \sigma = 0$.
- σ restricts to a Poisson structure on $D_{2k}(\sigma)$, of rank $\leq 2k$.

Bondal's conjecture

Conjecture (Bondal)

Let X be a Fano manifold and σ a Poisson structure on X . Then $D_{2k}(\sigma)$ has an irreducible component of dimension at least $2k + 1$, for every integer k with $0 \leq 2k < \dim X$.

- Stated by Bondal in 1993, in connection with non-commutative deformations of $\mathbb{P}^n$.
- For $\dim X = 2n$ and $k = n - 1$, the locus $D_{2n-2}(\sigma)$ is the zero divisor of $\sigma^n \in H^0(X, -K_X)$, which is nonzero on a Fano manifold. ✓
- The conjecture asks for the same behaviour at every k .

Positive results

Let $2r$ be the generic rank of σ .

- (1) **Polishchuk.** For $\dim X = 2n + 1$ odd, Bondal's conjecture holds for the **maximal** degeneracy locus $D_{2n-2}(\sigma)$.
- (2) **Beauville.** Polishchuk's argument applies verbatim to $D_{2r-2}(\sigma)$, in any dimension.
- (3) For $\dim X = 2n$ and σ of generic rank $2n$, the locus $D_{2n-2}(\sigma)$ is the divisor of zeros of $\sigma^n \in H^0(X, -K_X)$. **Gualtieri–Pym** settled the **submaximal** locus $D_{2n-4}(\sigma)$.

Together these give Bondal's conjecture for all Fano manifolds of dimension at most 4.

Main theorem

Theorem

Let X be a Fano manifold of dimension $2n + 1$ with $n > 1$, and let σ be a Poisson structure on X . Then $D_{2n-4}(\sigma)$ has an irreducible component of dimension at least $2n - 3$.

This is the **submaximal** degeneracy locus in odd dimension.

Corollary

Bondal's conjecture holds for all Fano manifolds of dimension five.

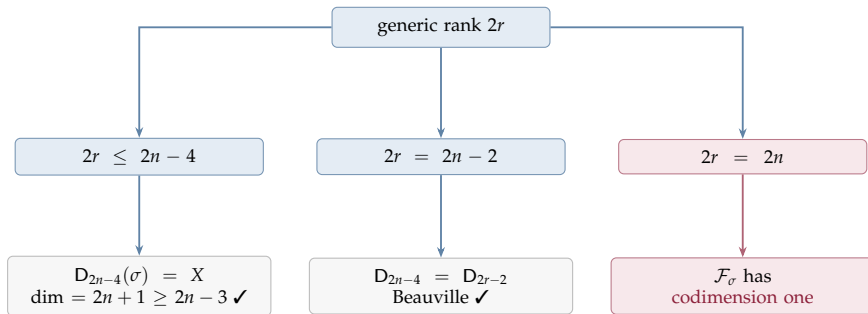
Status by dimension

dim X	D_0	D_2	D_4	D_6
2	anticanon.			
3	Polishchuk	$= X$		
4	Gualtieri–Pym	anticanon.		
5	new	Polishchuk	$= X$	
6	open	Gualtieri–Pym	anticanon.	
7	open	new	Polishchuk	$= X$

Throughout the table σ has *maximal* generic rank $2\lfloor \frac{1}{2} \dim X \rfloor$.

Reduction to maximal rank

Let $\dim X = 2n + 1$ and let $2r$ be the generic rank of σ .



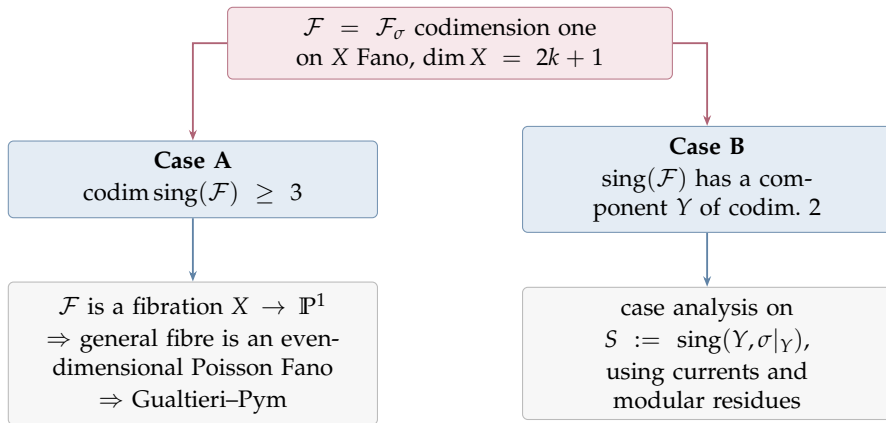
From now on

$\dim X = 2k + 1, k > 1$, σ of **maximal rank $2k$** , and $\mathcal{F} := \mathcal{F}_\sigma$ is a **codimension-one holomorphic foliation** on the Fano manifold X .

Goal: $D_{2k-4}(\sigma)$ has a component of dimension $\geq 2k - 3$.

The dichotomy

$\mathcal{F}$ is codimension one, so $\text{codim sing}(\mathcal{F}) \geq 2$ automatically.



Codimension-one foliations with small singular locus

X is a **weak Fano manifold** if $-K_X$ is nef and big. Druel proved that a regular foliation of any codimension on a weak Fano manifold is a fibration.

Theorem

Let X be a weak Fano projective manifold and $\mathcal{F}$ a codimension-one holomorphic foliation on X with $\text{codim sing}(\mathcal{F}) \geq 3$. Then $\mathcal{F}$ is defined by a morphism

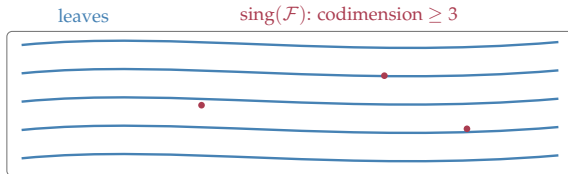
$$f: X \longrightarrow \mathbb{P}^1$$

with irreducible fibres.

- No Poisson geometry enters; this is what we use in Case A.
- The hypothesis $\text{codim sing}(\mathcal{F}) \geq 3$ is what the Bott connection needs.

The Bott connection under a small singular locus

For $\mathcal{F}$ regular, $\mathcal{N}_{\mathcal{F}}$ carries a flat partial connection along $\mathcal{T}_{\mathcal{F}}$, so $c_1(\mathcal{N}_{\mathcal{F}})^q$ lifts along $H^q(X, \det \mathcal{N}_{\mathcal{F}}^{\vee}) \rightarrow H^q(X, \Omega_X^q)$.



Lemma (Druel)

If $\text{codim } \text{sing}(\mathcal{F}) \geq q + 2$, then $c_1(\mathcal{N}_{\mathcal{F}})^q \in H^q(X, \Omega_X^q)$ lies in the image of

$$H^q(X, \det \mathcal{N}_{\mathcal{F}}^{\vee}) \longrightarrow H^q(X, \Omega_X^q).$$

Cohomology of locally free sheaves is unchanged by removing a closed subvariety of large enough codimension; work on X° and extend.

Curves and the normal bundle

Take $q = 1$ and $\text{codim sing}(\mathcal{F}) \geq 3$.

Tangent curves

If $\phi: C \rightarrow X$ is a compact curve **tangent** to $\mathcal{F}$, then $\phi^*c_1(\mathcal{N}_{\mathcal{F}}) = 0$ in $H^1(C, \Omega_C^1)$; in particular

$$\mathcal{N}_{\mathcal{F}} \cdot C = 0.$$

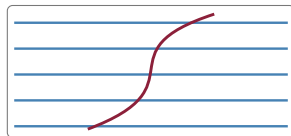


C tangent: $\mathcal{N}_{\mathcal{F}} \cdot C = 0$

Transverse rational curves

If $\phi: C \rightarrow X$ is a **rational** curve not tangent to $\mathcal{F}$, then

$$\mathcal{N}_{\mathcal{F}} \cdot C \geq 2.$$



$C \cong \mathbb{P}^1$ transverse: $\mathcal{N}_{\mathcal{F}} \cdot C \geq 2$

$\phi^*\mathcal{N}_{\mathcal{F}}^{\vee} \rightarrow \Omega_C^1$ is nonzero, dually $\mathcal{T}_C \rightarrow \phi^*\mathcal{N}_{\mathcal{F}}$ is nonzero, and $\deg \mathcal{T}_C = 2$.

Nefness of the normal bundle

Lemma

If X is weak Fano, then $\overline{\text{NE}}(X)$ is generated by classes of rational curves.

By Kodaira's lemma pick an effective E with $-K_X - \varepsilon E$ ample for $\varepsilon \in]0, 1]$; for small rational ε the pair $(X, \varepsilon E)$ is log Fano, and the cone theorem for klt pairs applies.



For every $a > 0$ the divisor $a\mathcal{N}_{\mathcal{F}} - K_X$ is nef and big. By the base point free theorem some $\mathcal{N}_{\mathcal{F}}^{\otimes m}$ is globally generated, so there is a surjective morphism with connected fibres

$$f: X \longrightarrow Y, \quad \mathcal{N}_{\mathcal{F}}^{\otimes m} \simeq f^* \mathcal{A}, \quad \mathcal{A} \text{ ample on } Y.$$

The base of the fibration

$\dim Y \leq 1$

By Baum–Bott, $c_1(\mathcal{N}_{\mathcal{F}})^2$ is represented by a cycle supported on the **codimension-two** components of $\text{sing}(\mathcal{F})$. There are none, so $c_1(\mathcal{N}_{\mathcal{F}})^2 = 0$. If $\dim Y \geq 2$ then $c_1(\mathcal{N}_{\mathcal{F}})^2 = \frac{1}{m^2} f^* c_1(\mathcal{A})^2 \neq 0$.

$\dim Y = 1$

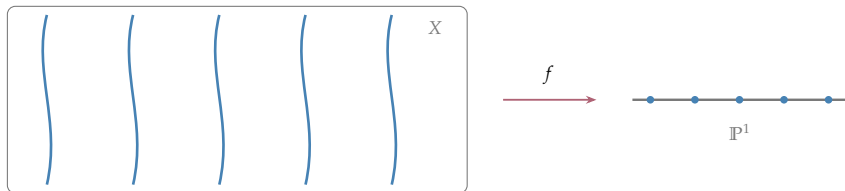
Let F be a general fibre. By adjunction $K_F = K_X|_F$, so $-K_F$ is nef and big and F is weak Fano, hence simply connected by Zhang. Also $\mathcal{N}_{\mathcal{F}}|_F$ is torsion, so $\mathcal{N}_{\mathcal{F}}|_F \simeq \mathcal{O}_F$ and $\omega|_F = 0$: F is tangent to $\mathcal{F}$, whence $\dim F \leq \dim X - 1$ and $\dim Y \geq 1$.

$Y \cong \mathbb{P}^1$, with irreducible fibres

$\dim Y = 1$ and Y normal, so Y is smooth. Connected fibres give $H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(X, \mathcal{O}_X) = 0$, so $Y \cong \mathbb{P}^1$. The fibres are tangent to $\mathcal{F}$, so their singular loci lie in $\text{sing}(\mathcal{F})$, of codimension ≥ 3 ; hence they are irreducible. $\square$

Case A

Assume $\text{codim sing}(\mathcal{F}_\sigma) \geq 3$, with $\dim X = 2k + 1$ Fano and σ of maximal rank $2k$.



fibres F : Poisson subvarieties, $\dim F = 2k$

- $\text{codim sing}(\mathcal{F}_\sigma) \geq 3$, so $\mathcal{F}_\sigma = \mathcal{T}_{X/\mathbb{P}^1}$ for a morphism $f: X \rightarrow \mathbb{P}^1$ with irreducible fibres.
- A general fibre F is a Poisson subvariety of codimension one, of **even** dimension $2k$, and $K_F = K_X|_F$ makes it Fano.
- $\sigma|_F$ has maximal rank $2k$, so Gualtieri–Pym’s submaximal theorem in even dimension gives a component of $D_{2k-4}(\sigma|_F)$ of dimension $\geq 2k - 3$.
- $D_{2k-4}(\sigma|_F) \subseteq D_{2k-4}(\sigma) \cap F$.



Pfaff fields

Definition (Esteves–Kleiman)

A **Pfaff field of degree p** on X is a pair $(\mathcal{L}, \eta)$ with $\mathcal{L}$ invertible and $\eta: \Omega_X^p \rightarrow \mathcal{L}$ an $\mathcal{O}_X$ -linear map.

Equivalently, for X smooth of dimension n ,

$$\eta \in H^0(X, \wedge^p \mathcal{T}_X \otimes \mathcal{L}) \cong H^0(X, \Omega_X^{n-p} \otimes \omega_X^\vee \otimes \mathcal{L}),$$

and $\ker \eta := \{v \in \mathcal{T}_X : \eta \wedge v = 0\}$ is a distribution.

Definition

$Y \subset X$ is **η -invariant** if $\eta|_Y$ factors through the surjection $\Omega_X^p|_Y \twoheadrightarrow \Omega_Y^p$.

Esteves and Kleiman used constraints of this kind to bound solutions of Pfaff equations. We need a version valid on compact Kähler manifolds.

$$\begin{array}{ccc} \Omega_X^p & \xrightarrow{\eta} & \mathcal{L} \\ \downarrow & & \downarrow \\ \Omega_Y^p & \xrightarrow[\eta_Y]{\text{---}} & \mathcal{L}|_Y \end{array}$$

Strongly directed positive currents

Definition

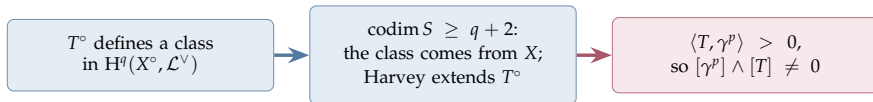
Let $\alpha \in H^0(X, \Omega_X^q \otimes \mathcal{L})$. A closed positive current T of bidegree (q, q) is **strongly directed by α** if locally

$$T = (\sqrt{-1})^{q^2} a \omega \wedge \bar{\omega},$$

with a a positive locally finite Borel measure and ω a local generator of $\alpha(\mathcal{L}^\vee) \subseteq \Omega_X^q$.

Theorem

Let X be compact Kähler of dimension n and let $\eta: \Omega_X^p \rightarrow \omega_X \otimes \mathcal{L}$ be the Pfaff field induced by α , $p = n - q$. Let $S \subset X$ be closed analytic with $\text{codim } S \geq q + 2$. If $X \setminus S$ carries a nonzero closed positive current strongly directed by α , then $H^p(\eta): H^p(X, \Omega_X^p) \rightarrow H^p(X, \omega_X \otimes \mathcal{L})$ is nonzero.



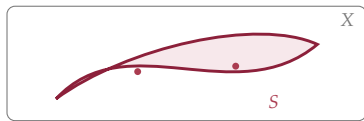
Invariant subvarieties

Proposition

Let X be compact Kähler of dimension n and $\eta: \Omega_X^p \rightarrow \mathcal{L}$ a Pfaff field. Let $S \subset Y \subset X$ be closed subvarieties such that

1. Y is η -invariant, irreducible, of dimension p ;
2. $\dim S \leq p - 2$;
3. $Y^\circ := Y \setminus S$ is smooth;
4. $\eta|_{Y^\circ}$ is nowhere vanishing.

Then $H^p(\eta): H^p(X, \Omega_X^p) \rightarrow H^p(X, \mathcal{L})$ is nonzero.



$Y, \dim Y = p$

Take T° to be the current of integration along Y° : it is closed, positive, and strongly directed by the twisted q -form induced by η , precisely because $\eta|_{Y^\circ}$ does not vanish.

Symplectic leaves on Fano manifolds

Proposition

Let X be compact Kähler and σ a holomorphic Poisson structure on X . Suppose X contains closed subvarieties $S \subset Y \subset X$ such that $Y \setminus S$ is a symplectic leaf and $\text{codim}(S, Y) \geq 2$. Then

$$h^{0, \dim Y}(X) > 0.$$

- This extends an earlier result of ours from compact symplectic leaves to leaf closures whose boundary is small.
- On a Fano manifold $h^{0,j}(X) = 0$ for $j > 0$. So a Fano Poisson manifold carries no algebraic symplectic leaf closure with small boundary.

The modular vector field

Let μ be a local trivialization of ω_X over $U \subset X$.

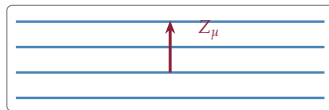
Definition

The **modular vector field** of σ with respect to μ is the unique Z_μ with

$$\iota_{Z_\mu} \mu = -d(\iota_\sigma \mu).$$

- Z_μ is a Poisson vector field: $\mathcal{L}_{Z_\mu} \sigma = 0$.
- $\mu \rightsquigarrow g\mu$ changes Z_μ by a Hamiltonian field, so its class is Weinstein's modular class.
- Along a leaf, Z_μ measures the failure of the leafwise Liouville measure to be preserved.

symplectic leaves



Modular residues

Definition (Gualtieri–Pym)

For $k \geq 0$, the **k -th modular residue** is a global section

$$\mathrm{Res}_k^{\mathrm{mod}}(\sigma) \in H^0\left(D_{2k}(\sigma), \mathcal{X}_{D_{2k}(\sigma)}^{2k+1}\right), \quad \mathcal{X}_D^\bullet := (\Omega_D^\bullet)^\vee,$$

determined locally by

$$i_* \mathrm{Res}_k^{\mathrm{mod}}(\sigma|_U) = (Z_\mu \wedge \sigma^k)|_{D_{2k}(\sigma) \cap U}, \quad i: D_{2k}(\sigma) \hookrightarrow X.$$



$\mathrm{Res}_k^{\mathrm{mod}}(\sigma)$ is a Pfaff field on $D_{2k}(\sigma)$.

Two constraints

Lemma (Gualtieri–Pym)

Let (X, σ) be a Poisson Fano manifold, $k \geq 1$, and Y an irreducible, closed, strong Poisson subscheme of $D_{2k}(\sigma)$ such that

1. $Y \not\subset D_{2k-2}(\sigma)$, and
2. Y is contained in the vanishing locus of $i_* \text{Res}_k^{\text{mod}}(\sigma) \in H^0(D_{2k}(\sigma), \wedge^{2k+1} \mathcal{T}_X|_{D_{2k}(\sigma)})$.

Then $\text{sing}(Y, \sigma|_Y)$ is nonempty and has a component of dimension at least $2k - 1$.

Proposition (Hodge constraint)

Let (X, σ) be compact Kähler Poisson and $Y \subset X$ a closed Poisson subvariety with

1. $\sigma|_Y$ generically of rank $2k$;
2. $\dim \text{sing}(Y, \sigma|_Y) \leq 2k - 2$;
3. the symplectic foliation on $Y \setminus \text{sing}(Y, \sigma|_Y)$ supports a closed strongly directed positive current, which is automatic if $\dim Y = 2k$.

Then $h^{0,2k}(X) > 0$.

Here $\text{sing}(Y, \sigma) := \text{sing}(Y) \cup (D_{2k-2}(\sigma) \cap Y)$ for Y Poisson of maximal rank $2k$.

Setting up Case B

$\dim X = 2k + 1$ Fano, $k > 1$, σ of maximal rank $2k$, $\mathcal{F} = \mathcal{F}_\sigma$ codimension one.

Assume $\text{sing}(\mathcal{F})$ has an irreducible component Y of codimension two, so $\dim Y = 2k - 1$, with the reduced structure.

Off $D_{2k-2}(\sigma)$ the sheaf $\sigma^\#(\Omega_X^1)$ is already a subbundle, so $\text{sing}(\mathcal{F}) \subseteq D_{2k-2}(\sigma)$, and in particular $Y \subseteq D_{2k-2}(\sigma)$.

Since $\dim Y = 2k - 1$ is **odd**, the maximal rank of $\sigma|_Y$ is exactly $2k - 2$, and

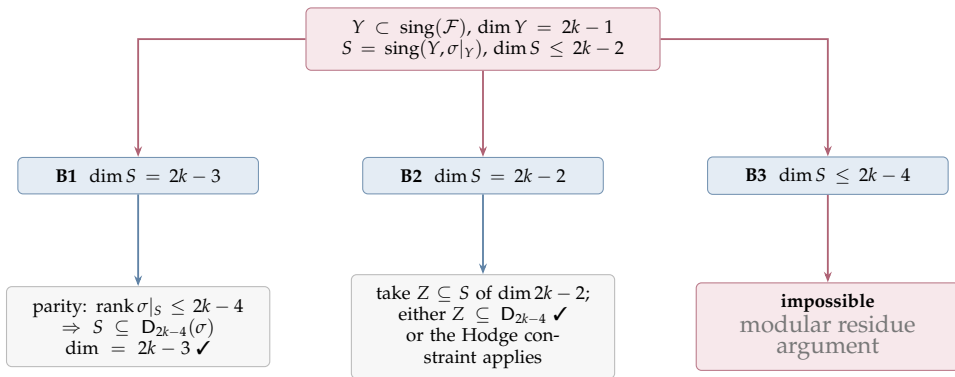
$$S := \text{sing}(Y, \sigma|_Y) = \text{sing}(Y) \cup (D_{2k-4}(\sigma) \cap Y), \quad \dim S \leq 2k - 2.$$

S and Y are strong Poisson subvarieties of $D_{2k-2}(\sigma)$.

Parity

If W is a Poisson subvariety of odd dimension $2j - 1$, then $\text{rank } \sigma|_W \leq 2j - 2$, that is, $W \subseteq D_{2j-2}(\sigma)$.

Case B: the trichotomy on $\dim S$



Goal: a component of $D_{2k-4}(\sigma)$ of dimension $\geq 2k - 3$.

Case B2

$\dim S = 2k - 2$. Let Z be an irreducible component of S of dimension $2k - 2$.

- If the generic rank of σ along Z is $< 2k - 2$, then $Z \subseteq D_{2k-4}(\sigma)$ and $\dim Z = 2k - 2 \geq 2k - 3$. ✓
- Otherwise $\sigma|_Z$ has maximal rank $2k - 2 = \dim Z$, so Z is the closure of an **algebraic symplectic leaf** of dimension $2k - 2$.
- X is Fano, so $h^{0,2k-2}(X) = 0$. In the Hodge constraint, hypothesis (2) must therefore fail:

$$\dim \operatorname{sing}(Z, \sigma|_Z) \geq 2k - 3.$$

- But $\operatorname{sing}(Z, \sigma|_Z)$ is a proper closed subset of the irreducible Z , so its dimension is at most $2k - 3$, hence exactly $2k - 3$. This is **odd**, so by parity

$$\operatorname{sing}(Z, \sigma|_Z) \subseteq D_{2k-4}(\sigma), \quad \dim = 2k - 3. \quad \checkmark$$

Case B3 is impossible, I

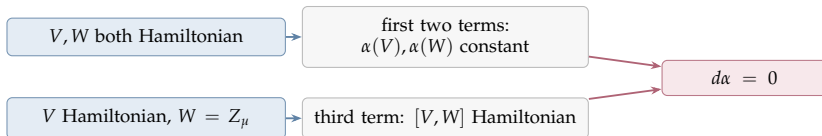
Suppose $\dim S \leq 2k - 4$. Then $Y^\circ := Y \setminus S$ is smooth of dimension $2k - 1$ and $\sigma|_{Y^\circ}$ is regular of rank $2k - 2$: the symplectic foliation of Y° is a regular codimension-one foliation. Let $\rho := \text{Res}_{k-1}^{\text{mod}}(\sigma)|_{Y^\circ}$, a section of $-K_{Y^\circ}$.

Claim: ρ vanishes somewhere.

Suppose not. Then ρ^{-1} is a nonvanishing holomorphic volume form on Y° , and $\alpha := \iota_{\sigma^{k-1}} \rho^{-1}$ is a nonvanishing holomorphic 1-form defining the symplectic foliation of Y° .

α vanishes on Hamiltonian vector fields, and $\alpha(Z_\mu)$ is a nonzero constant. Apply

$$d\alpha(V, W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]).$$



Case B3 is impossible, II

So $\sqrt{-1} \alpha \wedge \bar{\alpha}$ is a closed strongly directed positive current of bidegree $(1, 1)$ on Y° , and the Hodge constraint gives $h^{0, 2k-2}(X) > 0$, contradicting X Fano. Hence ρ does vanish somewhere.

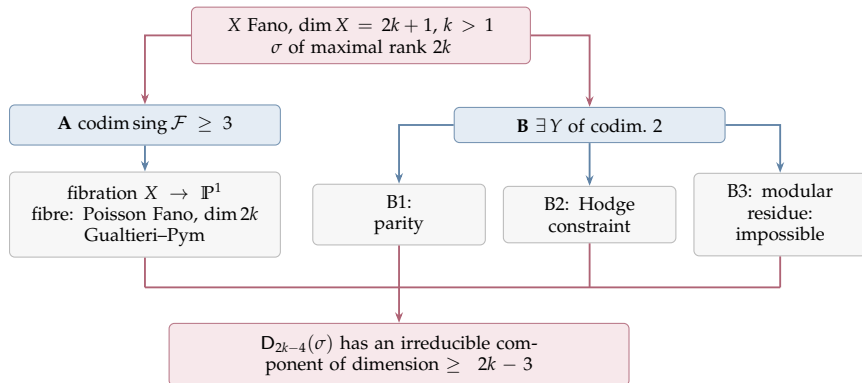
Set $\bar{\rho} := \text{Res}_{k-1}^{\text{mod}}(\sigma)|_Y$.

- By Gualtieri–Pym, the scheme-theoretic vanishing locus of $\bar{\rho}$, and each of its reduced irreducible components, are strong Poisson subschemes of $D_{2k-2}(\sigma)$.
- One may choose such a component W with $W \cap Y^\circ \neq \emptyset$:

$$W = Y \text{ (if } \rho \equiv 0), \quad \text{or} \quad \dim W = 2k - 2.$$

- In both cases $\text{sing}(W, \sigma|_W)$ is a strong Poisson subscheme contained in S .
- The Gualtieri–Pym lemma, applied to W with the integer $k - 1$, produces a component of $\text{sing}(W, \sigma|_W)$, hence of S , of dimension at least $2(k - 1) - 1 = 2k - 3$.
- This contradicts $\dim S \leq 2k - 4$. □

The proof, assembled



With $k = 2$: every Poisson structure on a Fano fivefold has $D_0(\sigma) \neq \emptyset$, with a component of dimension ≥ 1 . Together with Polishchuk's case D_2 , Bondal's conjecture holds in dimension five.